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Copositive Polynomial Approximation

ELI PASSOW AND LOUIS RAYMON

Mathematics Department, Temple University, Philadelphia, Pennsylvania 19122

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1. INTRODUCTION

There has been much interest in recent years in approximating functions by polynomials which are subject to certain constraints. Included are the problems of monotone approximation [1, 3, 5, 7, 11], comonotone approximation [4–6, 8], restricted range approximation [9, 10, 12–14], and others. In this paper we consider a new type of restriction.

Let P_n be the set of all algebraic polynomials of degree $\leqslant n$. Let $f \in C[-1, 1]$ and let $E_n(f) = \inf\{\| f - p \| : p \in P_n\}$ (sup norm on $[-1, 1]$). $E_n(f)$ is called the *degree of approximation* of f. Jackson's Theorem [2, p. 65] states that there exists a constant $C_1 > 0$ such that $E_n(f) \leqslant C_1\omega(f; 1/n)$, where ω is the modulus of continuity of f. If we restrict the approximating polynomials in some way, then we arrive at a problem of constrained approximation. For example, let $f \in C[-1, 1]$ be a function having a finite number of local extrema. Such a function is said to be *piecewise monotone*. The local extrema are called the *peaks* of f. Two functions are said to be *comonotone* on $[-1, 1]$ if they increase and decrease simultaneously on $[-1, 1]$. Let

$$E_n^*(f) = \inf\{\| f - p \| : p \in P_n , p \text{ comonotone with } f\}.$$

$E_n^*(f)$ is called the *degree of comonotone approximation* of f.

Estimates on $E_n^*(f)$ have been obtained by Passow, Raymon, and Roulier [6] and by Passow and Raymon [5], but they fall short of the Jackson estimate. Newman, Passow, and Raymon [4] have obtained results of a modified nature, as follows.

DEFINITION. Let f be piecewise monotone on $[-1, 1]$ with peaks at $-1 = x_1 < x_2 < \cdots < x_m = 1$. Let $\Delta = \frac{1}{2}\min_i (x_{i+1} - x_i)$. A sequence of polynomials $\{p_n\}$ is said to be *nearly comonotone* with f on $[-1, 1]$, if, for every ϵ satisfying $0 < \epsilon < \Delta$, p_n is comonotone with f on $[x_i + \epsilon, x_{i+1} - \epsilon]$, $i = 1, 2, \ldots, m - 1$, for n sufficiently large.

DEFINITION. A piecewise monotone function f will be called *proper piecewise monotone* if it satisfies the following: for any $\epsilon > 0$ and two successive peaks x_i, x_{i+1} of f there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \geqslant \delta \tag{1}$$

for all x, y in $[x_i + \epsilon, x_{i+1} - \epsilon]$, $x \neq y$.

LEMMA [4, p. 471]. *Let f be a proper piecewise monotone function on $[-1, 1]$ such that $f \in Lip_M 1$; i.e., $\omega(f; \delta) \leqslant M\delta$. Then there is a sequence $\{p_n\}$, $p_n \in P_n$, nearly comonotone with f, such that*

$$\| f - p_n \| \leqslant C_2 \omega(f; 1/n) \leqslant C_2 M/n.$$

Other results on nearly comonotone approximation have been obtained by Roulier [11], who showed, in particular, that if $f \in C^1[-1, 1]$ then the sequence of best approximations to f is nearly comonotone with f.

2. COPOSITIVE APPROXIMATION

f and g are said to be *copositive* on $[-1, 1]$ if $f(x)\, g(x) \geqslant 0$ for all $x \in [-1, 1]$. Let $f \in C[-1, 1]$ and let $\bar{E}_n(f) = \inf\{\| f - p \| : p \in P_n,\ p \text{ copositive with } f\}$. $\bar{E}_n(f)$ is called the *degree of copositive approximation* of f.

THEOREM 1. *Let $f \in C[-1, 1]$ be a proper piecewise monotone function, such that $f \in Lip_M 1$. Let f have peaks at $-1 = x_1 < x_2 < \cdots < x_k = 1$, and suppose that $f(x_i) \neq 0$, $i = 1, 2,..., k$. Then $\bar{E}_n(f) \leqslant d\omega(f; 1/n) \leqslant dM/n$, where d depends on f but not on n.*

Proof. Let $y_0 < y_1 < \cdots < y_l$ be the zeros of f in $[-1, 1]$. We assume, without loss of generality, that $M = 1$. Let

$$m = \tfrac{1}{4} \min\{\min_j | f(x_j)|, \min_j (x_{j+1} - x_j), \min_{i,j} | y_i - x_j |\}.$$

By the lemma, for n sufficiently large, there exists $p \in P_n$ such that

(i) p is comonotone with f on $[x_j + m, x_{j+1} - m]$, $j = 1, 2,..., k - 1$;
(ii) $\| f - p \| \leqslant C_2/n \leqslant m$.

On $I_j = [x_j - m, x_j + m]$, $m \geqslant | f(x_j) - f(x)| \geqslant | f(x_j)| - | f(x)| \geqslant 4m - | f(x)|$. Therefore, $| f(x)| \geqslant 3m$ on I_j. Since $| f(x) - p(x)| \leqslant m$, we have

$$| p(x)| \geqslant 2m > 0 \text{ on } I_j.$$

Thus $p(x) \neq 0$ for $x \in I_j$, $j = 0, 1,\ldots, k$, so that p has exactly $l + 1$ zeros in $[-1, 1]$, denoted by $y_0^*, y_1^*,\ldots, y_l^*$, where y_i and y_i^* are in the same interval $(x_j + m, x_{j+1} - m)$. p is thus a "nearly copositive" approximation to f. We now perturb p to obtain the desired polynomial.

Let $\delta = \delta(m)$ in the definition of a proper piecewise monotone function (1). Thus, $|f(y_i^*) - f(y_i)| \geqslant \delta\, |y_i^* - y_i|$, $i = 0, 1,\ldots, l$, so that

$$|y_i^* - y_i| \leqslant (1/\delta)|f(y_i^*) - f(y_i)| = (1/\delta)|f(y_i^*) - p(y_i^*)| \leqslant C_2/\delta n, \quad (2)$$

by the lemma.

Let $H_i(x) = \prod_{j=0;j\neq i}^{l} (x - y_j)/(y_i - y_j)$, and let $q(x) = \sum_{i=0}^{l} y_i^* H_i(x)$. $q \in P_l$ is thus the Lagrange interpolating polynomial satisfying

$$q(y_i) = y_i^*, \qquad i = 0, 1,\ldots, l. \quad (3)$$

Now $q(x) = \sum_{i=0}^{l} (y_i + \epsilon_i) H_i(x)$, where $|\epsilon_i| \leqslant C_2/\delta n$, from (2). Thus $q(x) = \sum_{i=0}^{l} y_i H_i(x) + \sum_{i=0}^{l} \epsilon_i H_i(x) = x + \sum_{i=0}^{l} \epsilon_i H_i(x)$. Hence,

$$\max_{-1\leqslant x\leqslant 1} |q(x) - x| \leqslant (C_2/\delta n) \sum_{i=0}^{l} |H_i(x)| \leqslant (C_l/\delta n), \quad (4)$$

where C_l depends on $y_0, y_1,\ldots, y_l$.

Also, $q'(x) = 1 + \sum_{i=0}^{l} \epsilon_i H_i'(x)$. Therefore, $q'(x) \geqslant 1 - \sum_{i=0}^{l} |\epsilon_i H_i'(x)| \geqslant 1 - b_l/n$ on $[-1, 1]$. Hence $q'(x) > 0$ for n sufficiently large. Thus, for n sufficiently large, q is a monotone approximation to x which satisfies $q(y_i) = y_i^*$, $i = 0, 1,\ldots, l$. Now let $s(x) = p(q(x))$. Then $s \in P_{nl}$ and $s(y_i) = p(q(y_i)) = p(y_i^*)$, by (3). Thus $s(y_i) = 0$, $i = 0, 1,\ldots, l$, and s has no other zeros in $[-1, 1]$. Hence s is copositive with f.

Also, $s'(x) = p'(q(x))\, q'(x)$, so that $\operatorname{sgn} s'(x) = \operatorname{sgn} p'(q(x))$. Thus s is nearly comonotone with f.

Finally,

$$\|f - s\| = \|f - p(q)\| \leqslant \|f - p\| + \|p - p(q)\|. \quad (5)$$

Now, $\|f - p\| \leqslant C_2/n$ by (ii). Also,

$$|p(x) - p(q(x))| \leqslant \omega(p; |x - q(x)|) \leqslant \omega(p; C_l/\delta n), \text{ by (4)}. \quad (6)$$

Now $\omega(p; h) = \sup_{|x-y|\leqslant h} |p(x) - p(y)| \leqslant \sup_{|x-y|\leqslant h} [|p(x) - f(x)| + |f(x) - f(y)| + |f(y) - p(y)|]$.

Thus,

$$\omega(p; h) \leqslant 2C_2/n + h, \qquad \text{by (ii) and the fact that } f \in \mathrm{Lip}_1\, 1. \quad (7)$$

Therefore, from (5)–(7),

$$\|f - s\| \leqslant C_2/n + 2C_2/n + C_l/\delta n = A_l/n.$$

Hence $\bar{E}_{nl}(f) \leqslant A_l/n$, so that $\bar{E}_n(f) \leqslant d/n$, where d depends upon f.

3. Best Copositive Approximation

In this section we prove that a best nth degree copositive approximation to a continuous function is unique.

Let u and 1 be defined on $[-1, 1]$, with $1(x) \leqslant u(x)$, and let $E_n(f; u, 1) = \inf\{\|f - p\| : p \in P_n, 1(x) \leqslant p(x) \leqslant u(x)$ for all $x \in [-1, 1]\}$ be the *degree of restricted range approximation* of f, relative to u and 1. Approximation of this type has been considered by Taylor [12–14], Schumaker and Taylor [10], and Roulier and Taylor [9]. In [10] it was shown that the best nth degree restricted range approximation is unique if $f \in C[-1, 1]$ and

$$1(x) \leqslant f(x) \leqslant u(x) \text{ for all } x \text{ in } [-1, 1].$$

We will show that copositive approximation can be viewed as a special case of restricted range approximation, through an appropriate choice of u and 1.

THEOREM 2. *Let $f \in C[-1, 1]$. Then the best* nth *degree copositive approximation to f is unique.*

Proof. If f has an infinitive number of sign changes then the only polynomial copositive with f is the zero polynomial, and, hence, the best copositive approximation is unique. We assume, therefore, that f has a finite number of sign changes.

Without loss of generality, $\|f\| \leqslant \frac{1}{2}$. Thus if p is an nth degree polynomial of best copositive approximation to f on $[-1, 1]$, then $\|p\| \leqslant 1$.

We now split $[-1, 1]$ into three types of subintervals:

Type I: $[a, b]$, where

(i) $f(x) \geqslant 0$ for all $x \in [a, b]$;
(ii) $f(x) \neq 0$ on any subinterval of $[a, b]$;
(iii) $f(a) = f(b) = 0$ (unless $a = -1$ or $b = 1$);
(iv) there is no subinterval properly containing $[a, b]$ with properties (i)–(iii).

Type II: $[c, d]$, where

(i′) $f(x) \leqslant 0$ for all $x \in [c, d]$;
(ii′) $f(x) \neq 0$ on any subinterval of $[c, d]$;
(iii′) $f(c) = f(d) = 0$ (unless $c = -1$ or $d = 1$);
(iv′) there is no subinterval properly containing $[c, d]$ with properties (i′)–(iii′).

Type III: $[e, g]$, where $f(x) = 0$ on $[e, g]$, but not on any subinterval properly containing [e,g].

We consider two cases.

Case 1. There are no subintervals of Type III; i.e., f does not vanish on any subinterval of $[-1, 1]$.

Let u and l be defined on $[-1, 1]$ as follows:

On an interval $[a, b]$ of Type I, let

$$u(x) = \begin{cases} \max(f(x), 2n^2(x - a)), & x \in [a, (a + b)/2], \\ \max(f(x), -2n^2(x - b)), & x \in ((a + b)/2, b], \end{cases}$$

and $\mathrm{l}(x) = 0$, $x \in [a, b]$.

On an interval $[c, d]$ of Type II, let

$$u(x) = 0, \qquad x \in [c, d]$$

and

$$l(x) = \begin{cases} \min(f(x), -2n^2(x - c)), & x \in [c, (c + d)/2], \\ \min(f(x), 2n^2(x - d)), & x \in ((c + d)/2, d]. \end{cases}$$

(If $a = -1$, let $u(x) = \max(f(x), -2n^2(x - b))$, $x \in [-1, b]$. Similar modifications of u are necessary if $b = 1$, and of l if $c = -1$ or $d = 1$.)

Let p be any nth degree polynomial such that $\| p \| \leqslant 1$. Then $\| p' \| \leqslant n^2$, by the Markov estimate on p' [2, p. 40]. Thus, by our choice of u and l, p is copositive with f if and only if $\mathrm{l}(x) \leqslant p(x) \leqslant u(x)$ for all $x \in [-1, 1]$. Hence, p is a best copositive approximation to f if and only if p is a best restricted range approximation to f, relative to u and l. But the latter is unique, by [10], and thus the theorem is proved in this case.

Case 2. There are intervals of Type III.

We must now modify our definition of u and l. Let $[e, g]$ be a subinterval of Type III. Let $[\alpha, \beta]$ be the largest subinterval of $[-1, 1]$ containing [e, g] and intervals of Types I and III exclusively. Let $[\gamma, \delta]$ be the largest subinterval of $[-1, 1]$ containing $[e, g]$ and intervals of Types II and III exclusively. Define

$$u(x) = \begin{cases} \max(f(x), 2n^2(x - \alpha)), & x \in [\alpha, (\alpha + \beta)/2], \\ \max(f(x), -2n^2(x - \beta)), & x \in ((\alpha + \beta)/2, \beta] \end{cases}$$

and

$$l(x) = \begin{cases} \min(f(x), -2n^2(x - \gamma)), & x \in [\gamma, (\gamma + \delta)/2], \\ \min(f(x), 2n^2(x - \delta)), & x \in ((\gamma + \delta)/2, \delta]. \end{cases}$$

On the remaining subintervals of Types I and II, u and l are defined as in Case 1. With these modifications the proof is completed as in the previous case.

Note added in proof. Theorem 1 can be extended to functions which do not belong to $\mathrm{Lip}_M 1$ for any M. By the usual method of preapproximating f by piecewise linear functions, copositive and comonotone with f [cf. R. P. Feinerman and D. J. Newman, "Polynomial Approximation," Williams and Wilkins, Baltimore, 1974, p. 38], we obtain the following estimates of $\bar{E}_n(f)$:

THEOREM 1'. *Let* $f \in C[-1, 1]$ *be a proper piecewise monotone function, with peaks at* $-1 = x_1 < x_2 < \cdots < x_k = 1$, *and suppose that* $f(x_i) \neq 0$, $i = 1, 2,..., k$. *Then* $\bar{E}_n(f) = d_1 \omega(f; 1/n)$, *where* d_1 *is independent of* n.

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